EPHOU 96-008

December 1996

On Explicit Evaluations Around the Conformal Point in $N = 2$ Supersymmetric Yang-Mills Theories

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Abstract

We show how to give the expression for periods, Higgs field and its dual of $N = 2$ supersymmetric Yang-Mills theory around the conformal point. This is achieved by evaluating the integral representation in the weak coupling region, and by using analytic continuation to the conformal point. The explicit representation is shown for the $SU(2)$ theory with matter fields and also for pure $SU(N)$ and pure $SO(2N)$ theory around the conformal point where the relation to the beta function of the theory is clarified. We also discuss a relation between the fixed points in the $SU(2)$ theories with matter fields and the Landau-Ginzburg point of 2-D $N = 2$ SCFT.

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1 Introduction

Recent years many progress about four dimentional $N = 2$ supersymmetric Yang-Mills theories have been made. Seiberg and Witten [1] solved the low energy effective theory in the $SU(2)$ theory without matter fields exactly, based on the duality and holomorphy by introducing the elliptic curve. Following this work various generalizations introducing the matter fields or gauge group being higher than $SU(2)$ have been investigated by many people [2, 3, 4, 5, 6, 7, 8, 9]. The solution of these theories extesively studied in the weak coupling region by various method, such as solving the Picard-Fuchs equation [11, 12], purtubative treatments to obtain the prepotential [13, 14]. The direct tests have been made by comparison with the instanton method in the case of $SU(2)$ theory with matter fields [15, 16], and even for the higher rank gauge groups [17], which provide the consistent results.

Apart from the analysis in the weak coupling region, the power of the exact results should be used in the analysis in the strong coupling region, where one finds truely non-perturbative results. Among the analysis of the strong coupling region, one of the striking fact is the existence of the conformal points [18, 19, 20] where the prepotentials have no dependence of the dynamical mass scale. These theories are classified by scaling behaviors around the conformal point [20]. It seems interesting to investigate the theories around this points by deriving the explicit form of fields, which seems to provide us of a more concrete behaviour of the critical theories.

In previous works [21, 22] we have evaluated the integral representation about the period, Higgs field and its dual on such situations. The results for $SU(2)$ Yang-Mills theory with matter fields [21] can be analytically continuated around the conformal point when the bare masses take the critical values. In this article, we generalize this approach to investigate the expression around the conformal point. We treat moduli parameters and bare masses as deviations from the conformal point. After evaluating the integral representation explicitely in the region where only one parameter is very large but other parameters are near the confomal point, we perform the analytic continuation of one large parameter to be near

the conformal point. By use of the analyic continuation we can get the expression around the conformal point. Usually, analytic continuation to the region where the logarithmic singularity exits must be treated with care because the result may depend on the choice of variables. In other words, some choice of variable are valid only within some branch. However, when we consider the analytic continuation to the critical point where there is no such singularity, the confirmation of such analytic continuation turns out to be easy, as we will see in each cases. As the matter of fact, this approach can be considered as a generalization of the one given to obtain the periods for Calabi-Yau systems [23]. Of course, there are a variety of the class of conformal point [20] so that we cannot exhaust all known cases. In this paper, we will deal with $SU(2)$ Yang-Mills theory with massive hypermultiplets, and $SU(N)$ and also $SO(2N)$ Yang-Mills theory without matters. We also provide expressions of Higgs field and its dual around the conformal point for the pure $SU(N)$ and also $SO(2N)$ Yang-Mills theories.

This article is organized as follows. In section 2, we will obtain the expression of the fields around conformal points in $SU(2)$ theory with matter fields case, and verify that the result recovers our previous result which was obtained by transformations of the hypergeometric functions[21] when the deviation of mass parameters from the conformal point are set to be zero. We will also discuss the relation to 2-D $N = 2$ SCFT through the simple correspodence to the deformation of curve on $W\mathbf{CP^2}$. This relation has been pointed out recently in the different context in the ref. [24]. In section 3, we will derive the form of Higgs field and its dual in the pure $SU(N)$ theory around the conformal point and clarify the relation to the beta function of theory. In section 4, we will study the pure $SO(2N)$ theory around the conformal point and discuss the validity of the expression. The last section will be devoted to some discussion.

2 $SU(2)$ Yang-Mills theories with matter fields

In this section we treat the $SU(2)$ theory with N_f matter fields ($N_f \leq 3$), to verify

that our approach recover the previous results which was obtained by transformations of the hypergeometric functions[21].

First of all we consider $N_f = 1$ case, whose curve of $N_f = 1$ theory is given by

$$y^2 = x^2(x-u) + \frac{1}{4}m\Lambda^3 x - \frac{\Lambda^6}{64}. \tag{2.1}$$

In order to calculate the periods:

$$\frac{\partial a}{\partial u} = \oint_\alpha \frac{dx}{y}, \quad \frac{\partial a_D}{\partial u} = \oint_\beta \frac{dx}{y}, \tag{2.2}$$

around the conformal point $u = \frac{3}{4}\Lambda^2,\ m = \frac{3}{4}\Lambda$ where the curve becomes degenerate as $y^2 = (x - \frac{\Lambda^2}{4})^3$, we introduce the deviations from this conformal point as $\tilde{u} = u - \frac{3}{4}\Lambda^2,\ \tilde{m} = m - \frac{3}{4}\Lambda$. The strategy is that after calculating the period in the weak coupling region $\tilde{u} \sim \infty$, we analytically continuate around the conformal point $\tilde{u} \sim 0$. It should be noted that similar consideration has been used to evaluate periods for Calabi-Yau manifolds [23]. Rescaling the variable as $x = \frac{1}{4}\Lambda^2 z$ the curve becomes

$$y^2 = (\frac{\Lambda^2}{4})^3(z-1)^3 - (\frac{\Lambda^2}{4})^2\tilde{u}z^2 - (\frac{\Lambda^2}{4})^2\Lambda m z, \tag{2.3}$$

and thus we consider $\frac{\tilde{u}}{\Lambda^2}$ and $\frac{\tilde{m}}{\Lambda}$ as perturbations from the conformal point. Expanding the period with respect to $1/\tilde{u}$ we have the expression for the period as follows:

$$\begin{aligned}\oint \frac{dx}{y} = \tilde{u}^{-1/2}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{1}{2}+s)\Gamma(s+1)\Gamma(-s)}{\Gamma(\frac{1}{2})\Gamma(s-m+1)m!}& \\ \times \oint dz\,(1-z)^{3(s-m)}z^{-2s+m-1}\left(\frac{\Lambda^2}{4\tilde{u}}\right)^s\left(\frac{\tilde{m}}{\Lambda}\right)^m&,\end{aligned} \tag{2.4}$$

where we have introduced Barnes-type integral representation. From this expression we can find that $\frac{\partial a}{\partial u}$ is obtained by picking up poles $z = 0$ along α cycle in the weak coupling region. In this way, we find $\frac{\partial a}{\partial u}$ in the weak coupling region is of the form

$$\begin{aligned}\frac{\partial a}{\partial u} &= \frac{\sqrt{3}}{\tilde{u}^{1/2}2\pi}\sum_{n,m=0}^{\infty}\frac{\Gamma(n+m+\frac{1}{2})\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{m}{2}+\frac{1}{2})\Gamma(n+\frac{m}{2}+1)n!m!}\left(-\frac{27\Lambda^2}{16\tilde{u}}\right)^n\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\ &= \frac{\sqrt{3}}{\tilde{u}^{1/2}2\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{m+1}{2})\Gamma(\frac{m}{2}+1)m!}\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\ &\qquad \times {}_3F_2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2}+m;\frac{m+1}{2},\frac{m}{2}+1;-\frac{27\Lambda^2}{16\tilde{u}}\right),\end{aligned} \tag{2.5}$$

where ${}_3F_2$ is the generalized hypergeometric function, which is defined as [25]

$$
{}_pF_{p-1}(a_1,\cdots,a_p;b_1,\cdots,b_{p-1};z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_{p-1})_n}\frac{z^n}{n!},\quad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.6}
$$

Integrating with respect to $\tilde{u}$ of (2.5) we have Higgs field a up to mass residue in the weak coupling region in the following form

$$
\begin{aligned}
a=\frac{\sqrt{3}\tilde{u}^{1/2}}{\pi}&\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{m+1}{2})\Gamma(\frac{m}{2}+1)m!}\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\
&\times {}_3F_2\left(\frac{1}{3},\frac{2}{3},m-\frac{1}{2};\frac{m+1}{2},\frac{m}{2}+1;-\frac{27\Lambda^2}{16\tilde{u}}\right).
\end{aligned} \tag{2.7}
$$

The analytic continuation from this expression to around the conformal point can be performed to obtain

$$
\begin{aligned}
a=\sqrt{\pi}\tilde{u}^{1/2}\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}&\sum_{m=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(m-\frac{5}{6})}{\Gamma(\frac{11}{6})\Gamma(-\frac{1}{6})\Gamma(m+\frac{1}{3})m!}\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m\right. \\
&\times {}_3F_2\left(\frac{1}{3},-\frac{m}{2}+\frac{5}{6},-\frac{m}{2}+\frac{1}{3};\frac{2}{3},-m+\frac{11}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right) \\
-\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}&\frac{\Gamma(\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(m-\frac{7}{6})}{\Gamma(-\frac{7}{6})\Gamma(\frac{13}{6})\Gamma(m-\frac{1}{3})m!}\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m \\
&\left.\times {}_3F_2\left(\frac{2}{3},-\frac{m}{2}+\frac{7}{6},-\frac{m}{2}+\frac{2}{3};\frac{4}{3},-m+\frac{13}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right)\right\}.
\end{aligned} \tag{2.8}
$$

If we set $\tilde{m}=0$ we can recover the previous result [21] where a is represented by the generalized hypergeometric function ${}_3F_2$ in terms of $\tilde{u}$.

Next we consider a_D. In this case we integrate (2.4) from $z=0$ to $z=1$ and evaluate double poles which give the logarithmic terms [22]. Quite similarly a_D in the weak coupling region can be written as

$$
\begin{aligned}
\frac{\partial a_D}{\partial u}=\frac{-1}{2(-)^{\frac{1}{2}}\tilde{u}^{\frac{1}{2}}\pi}&\sum_{n,m}\frac{\Gamma(n+m+\frac{1}{2})\Gamma(3n+1)}{\Gamma(\frac{1}{2})\Gamma(n+1)^2\Gamma(2n+m+1)}\left(\frac{\Lambda^2}{4\tilde{u}}\right)^n\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m \\
&\times\left[\psi(n+m+\frac{1}{2})+3\psi(3n+1)-2\psi(n+1)-2\psi(2n+m+1)+\ln\left(\frac{\Lambda^2}{4\tilde{u}}\right)\right]
\end{aligned} \tag{2.9}
$$

where $\psi(z)$ is defined by $\frac{d\Gamma(z)}{dz}=\psi(z)\Gamma(z)$. Analytic continuation to the region $\tilde{u}\sim 0$ and integration with respect to $\tilde{u}$ give the expression up to mass residue around the conformal

point as follows

$$
\begin{aligned}
a_D = & \frac{-\sqrt{\pi}\tilde{u}^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}}\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}\sum_{m=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(m-\frac{5}{6})}{\Gamma(\frac{11}{6})\Gamma(-\frac{1}{6})\Gamma(m+\frac{1}{3})m!}\left(\frac{\Lambda m}{4\tilde{u}}\right)^m\right. \\
& \qquad \times {}_3F_2\left(\frac{1}{3},-\frac{m}{2}+\frac{5}{6},-\frac{m}{2}+\frac{1}{3};\frac{2}{3},-m+\frac{11}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right) \\
& +\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}\frac{\Gamma(\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(m-\frac{7}{6})}{\Gamma(-\frac{7}{6})\Gamma(\frac{13}{6})\Gamma(m-\frac{1}{3})m!}\left(\frac{\Lambda m}{4\tilde{u}}\right)^m \\
& \qquad \left.\times {}_3F_2\left(\frac{2}{3},-\frac{m}{2}+\frac{7}{6},-\frac{m}{2}+\frac{2}{3};\frac{4}{3},-m+\frac{13}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right)\right\}.
\end{aligned} \tag{2.10}
$$

As the parameter approaching the point $\frac{\tilde{m}}{\tilde{u}} \to 0,\ \frac{\tilde{u}}{\Lambda^2} \to 0$, we find $a = a_D$, which implies that the theory is completely free theory. Therefore the conformal point is certainly the fixed point where the beta function of the theory vanishes. Since $a,\ a_D \sim \tilde{u}^{\frac{5}{6}}$ near the conformal point and $a,\ a_D$ are propotional to mass scale of the theory, the conformal dimention of $\tilde{u}$ is $\frac{6}{5}$, which has been observed in [19].

In the $N_f = 2$ theory, we use the curve of forth order:

$$
\begin{aligned}
y^2 &= (x^2 - u + \frac{\Lambda^2}{8})^2 - \Lambda^2(x+m_1)(x+m_2) && (2.11)\\
&= (x^2 - u + \frac{\Lambda^2}{8})^2 - \Lambda^2(x^2+Mx+N), && (2.12)
\end{aligned}
$$

where we introduce symmetrized mass parameters $M = m_1 + m_2$ and $N = m_1 m_2$. We shift the parameters from the conformal point as $\tilde{u} = u - \frac{3\Lambda^2}{8},\quad \tilde{M} = M - \Lambda,\quad \tilde{N} = N - \frac{\Lambda^2}{4}$, and rescale $x + \frac{\Lambda}{2} = \tilde{u}^{1/2}z$, we find that the curve can be written as

$$
y^2 = \tilde{u}^2(z^2-1)^2 - 2\Lambda\tilde{u}^{\frac{3}{2}}(z^3-z) - \Lambda^2\tilde{M}\tilde{u}^{\frac{1}{2}}z - \Lambda^2\tilde{N}', \tag{2.13}
$$

where $\tilde{N}' = \tilde{N} - \tilde{M}\Lambda/2$. As is the case of $N_f = 1$, we evaluate the period in the weak coupling region by expanding with respect to $1/\tilde{u}$ and mass parameters, and by picking up poles at $z = 1$ along the α cycle to find

$$
\begin{aligned}
\frac{\partial a}{\partial u} = & \frac{\tilde{u}^{-\frac{1}{2}}}{2\pi}\sum_{m,l=0}^{\infty}\frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(\beta_{l,m}+\frac{1}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+1)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m \\
& \times {}_4F_3\left(\alpha_{l,m}+\frac{1}{4},\alpha_{l,m}+\frac{3}{4},l+\frac{1}{2},\beta_{l,m}+\frac{1}{2};\frac{1}{2},2\alpha_{l,m}+\frac{1}{2},2\alpha_{l,m}+1;-\frac{\Lambda^2}{\tilde{u}}\right)
\end{aligned} \tag{2.14}
$$

$$+\frac{\tilde{u}^{-\frac{1}{2}}}{2\pi}\left(\frac{2\Lambda}{\tilde{u}^{\frac{1}{2}}}\right)\left(\frac{\Lambda^2\tilde{M}}{\tilde{u}^{\frac{3}{2}}}\right)\sum_{m,l=0}^{\infty}\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\beta_{l,m}+\frac{5}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+4)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m$$
$$\times{}_4F_3\left(\alpha_{l,m}+\frac{5}{4},\alpha_{l,m}+\frac{7}{4},l+\frac{3}{2},\beta_{l,m}+\frac{5}{2};\frac{3}{2},2\alpha_{l,m}+\frac{5}{2},2\alpha_{l,m}+2;-\frac{\Lambda^2}{\tilde{u}}\right),$$

where $\alpha_{l,m}=l+\frac{m}{2}$, $\beta_{l,m}=2m+3l$. In the $N_f=2$ theory, $\frac{\partial a}{\partial u}$ has a additional part which is propotional to $\tilde{M}$ and vanishes when $\tilde{M}=0$.

Next we consider $\frac{\partial a_D}{\partial u}$. Performing the line integral from $z=0$ to $z=1$ and evaluating double poles, we have $\frac{\partial a_D}{\partial u}$ in the weak coupling region as

$$\begin{aligned}\frac{\partial a_D}{\partial u}=&\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\sum_{l,m,n}\frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(\beta_{l,m}+\frac{1}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+1)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m\left(-\frac{\Lambda^2}{\tilde{u}}\right)^n\\
&\times\frac{(\alpha_{l,m}+\frac{1}{4})_n(\alpha_{l,m}+\frac{3}{4})_n(l+\frac{1}{2})_n(\beta_{l,m}+\frac{1}{2})_n}{(\frac{1}{2})_n(2\alpha_{l,m}+\frac{1}{2})_n(2\alpha_{l,m}+1)_n}\left[\ln\left(-\frac{\Lambda^2}{\tilde{u}}\right)\right.\\
&+\psi_n(\alpha_{l,m}+\frac{1}{4})+\psi_n(\alpha_{l,m}+\frac{3}{4})+\psi_n(\beta_{l,m}+\frac{1}{2})\\
&\left.+\psi_n(l+\frac{1}{2})-\psi_n(1)-\psi_n(\frac{1}{2})-\psi_n(2\alpha_{l,m}+\frac{1}{2})-\psi_n(2\alpha_{l,m}+1)\right]\\
&+\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\left(\frac{2\Lambda}{\tilde{u}}\right)\left(\frac{\Lambda^2\tilde{M}}{\tilde{u}^{\frac{3}{2}}}\right)\sum_{l,m,n}\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\beta_{l,m}+\frac{5}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+4)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m\\
&\times\left(-\frac{\Lambda^2}{\tilde{u}}\right)^n\frac{(\alpha_{l,m}+\frac{5}{4})_n(\alpha_{l,m}+\frac{7}{4})_n(l+\frac{3}{2})_n(\beta_{l,m}+\frac{5}{2})_n}{(\frac{3}{2})_n(2\alpha_{l,m}+\frac{5}{2})_n(2\alpha_{l,m}+2)_n}\left[\ln\left(-\frac{\Lambda^2}{\tilde{u}}\right)\right.\\
&+\psi_n(\alpha_{l,m}+\frac{5}{4})+\psi_n(\alpha_{l,m}+\frac{7}{4})+\psi_n(\beta_{l,m}+\frac{5}{2})\\
&\left.+\psi_n(l+\frac{3}{2})-\psi_n(1)-\psi_n(\frac{3}{2})-\psi_n(2\alpha_{l,m}+2)-\psi_n(2\alpha_{l,m}+\frac{5}{2})\right],\end{aligned}\tag{2.15}$$

where $\psi_n(\alpha)=\psi(n+\alpha)$.

Analytic continuation of $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ to the region $\tilde{u}\sim 0$ gives four kinds of ${}_4F_3$, and a and a_D are also represented by ${}_4F_3$ after integration with respect to $\tilde{u}$. By defining Φ as

$$\begin{aligned}\Phi(\delta,\epsilon;\rho,\sigma,\mu)=\ {}_4F_3\Big(&-\alpha_{l,m}+\delta,-\alpha_{l,m}+\delta+\frac{1}{2},\alpha_{l,m}+\epsilon,\alpha_{l,m}+\epsilon+\frac{1}{2}\\
&;\alpha_{l,m}-\beta_{l,m}+\rho,\frac{m}{2}+\sigma,\mu;-\frac{\tilde{u}}{\Lambda^2}\Big),\end{aligned}\tag{2.16}$$

and using this function, we find that a around the conformal point can be written in the form:

$$a=\frac{\sqrt{\pi}\tilde{u}^{\frac{1}{2}}}{2}\sum_{m,l}\left(-\frac{\Lambda^2\tilde{M}^2}{\tilde{u}^2}\right)^l\left(\frac{\Lambda\tilde{N}'}{2\tilde{u}^{\frac{3}{2}}}\right)^m\frac{1}{\Gamma(\frac{1}{2})\Gamma(m+1)\Gamma(2l+1)\sqrt{2}}$$

$$\times \left[c_1 \left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{1}{4}} \frac{\Gamma(-\frac{m}{2}+\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{1}{4})}{\Gamma(\alpha_{l,m}+\frac{3}{4})\Gamma(\frac{1}{4}-\alpha_{l,m})} \Phi\left(\frac{1}{4},\frac{1}{4};\frac{7}{4},\frac{3}{4},\frac{1}{2}\right) \right.$$
$$-c_2 \left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{3}{4}} \frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(-\frac{m}{2}-\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}-\frac{1}{4})}{\Gamma(\alpha_{l,m}+\frac{1}{4})\Gamma(-\alpha_{l,m}-\frac{1}{4})\Gamma(2\alpha_{l,m}-\frac{1}{2})} \Phi\left(\frac{3}{4},\frac{3}{4};\frac{9}{4},\frac{5}{4},\frac{3}{2}\right) \tag{2.17}$$
$$+c_3 \left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{3}{4}} \left(\frac{\Lambda^2\tilde{M}}{2\tilde{u}^{\frac{3}{2}}}\right) \frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\frac{1}{4}-\frac{m}{2})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{5}{4})}{\Gamma(\alpha_{l,m}+\frac{3}{4})\Gamma(\frac{1}{4}-\alpha_{l,m})\Gamma(2\alpha_{l,m}+\frac{3}{2})} \Phi\left(-\frac{1}{4},\frac{3}{4};\frac{3}{4},\frac{3}{4},\frac{1}{2}\right)$$
$$\left. -c_4 \left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{5}{4}} \left(\frac{\Lambda^2\tilde{M}}{2\tilde{u}^{\frac{3}{2}}}\right) \frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(-\frac{m}{2}-\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{3}{4})}{\Gamma(\alpha_{l,m}+\frac{1}{4})\Gamma(-\alpha_{l,m}-\frac{1}{4})\Gamma(2\alpha_{l,m}+\frac{3}{2})} \Phi\left(\frac{1}{4},\frac{5}{4};\frac{5}{4},\frac{5}{4},\frac{3}{2}\right) \right],$$

where $c_1 = c_2 = c_3 = c_4 = 1$. We find that the expression for a_D is given by changing c_i as $c_1 = c_3 = (-1)^m,\ c_2 = c_4 = -(-1)^m$. If we set $\tilde{M} = \tilde{N}' = 0$ we can recover the previous result [21]. As in the $N_f = 1$ theory, we see that the conformal point is the fixed point of this theory from the relation $a \sim a_D$ on this point. Reading the leading power of the expression (2.17), we see that the conformal dimension of $\tilde{u}$ is $\frac{4}{3}$ [19].

In the $N_f = 3$ theory, the curve is given by

$$\begin{aligned} y^2 &= (x^2 - u + \frac{\Lambda}{4}x + \frac{(m_1+m_2+m_3)\Lambda}{8})^2 - \Lambda(x+m_1)(x+m_2)(x+m_3) \\ &= (x^2 - u + \frac{\Lambda}{4}x + \frac{\Lambda L}{8})^2 - \Lambda(x^3 + Lx^2 + Mx + N), \end{aligned} \tag{2.18}$$

where $L = m_1 + m_2 + m_3,\quad M = m_1m_2 + m_2m_3 + m_3m_1,\quad N = m_1m_2m_3$. We shift the parameter from the conformal point as $u' = u - \frac{\Lambda^2}{32},\ \tilde{L} = L - \frac{3\Lambda}{8},\ \tilde{M} = M - \frac{3\Lambda^2}{64},\ \tilde{N} = N - \frac{\Lambda^3}{512}$, the curve becomes

$$y^2 = (x+\frac{\Lambda}{8})^3(x-\frac{7\Lambda}{8}) - 2(u' - \frac{\Lambda\tilde{L}}{8})(x+\frac{\Lambda}{8})^2 - \Lambda(\tilde{L}x^2 + \tilde{M}x + \tilde{N}) + (u' - \frac{\Lambda\tilde{L}}{8})^2 \tag{2.19}$$

Setting $\tilde{u} = u' - \frac{\Lambda\tilde{L}}{8}$ and rescaling $x + \frac{\Lambda}{8} = \tilde{u}^{1/2}z$, we find that the curve can be written as

$$y^2 = \tilde{u}^2(z^2-1)^2 - \tilde{u}^{\frac{3}{2}}\Lambda z^3 - \tilde{L}\Lambda\tilde{u}z^2 - \tilde{u}^{\frac{1}{2}}Az + B, \tag{2.20}$$

where $A = \Lambda\tilde{M} - \frac{\Lambda^2\tilde{L}}{4},\ B = \frac{\Lambda^2\tilde{L}^2}{64} + \frac{\Lambda^2\tilde{M}}{8} - \Lambda\tilde{N}$. Evaluation of the integral for the period and analytic continuation from $\tilde{u} \sim \infty$ to $\tilde{u} \sim 0$ are same as $N_f = 2$ case. In this way, we can obtain the period in the weak coupling region in the form:

$$\frac{\partial a}{\partial u} = \frac{\tilde{u}^{-\frac{1}{2}}}{2\sqrt{\pi}} \sum_{l,m,p,q}^{\infty} \frac{\Gamma(3\eta_{l,p}+\frac{1}{2})\Gamma(\omega_{l,p,q}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2\chi_{l,p,q}+1)l!m!(2p)!} \left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l \left(\frac{A^2}{\tilde{u}^3}\right)^p \left(\frac{B}{\tilde{u}^2}\right)^q$$

$$\times\ {}_4F_3\left(\eta_{l,p}+\frac{1}{6},\eta_{l,p}+\frac{1}{2},\eta_{l,p}+\frac{5}{6},\omega_{l,p,q}+\frac{1}{2}\right.$$
$$\left.;\frac{1}{2},\chi_{l,p,q}+\frac{1}{2},\chi_{l,p,q}+1;-\frac{27\Lambda^2}{256\tilde{u}}\right). \tag{2.21}$$

$$-\frac{\tilde{u}^{-\frac{1}{2}}}{2\sqrt{\pi}}\left(\frac{\Lambda A}{\tilde{u}^2}\right)\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{5}{2})\Gamma(\omega_{l,p,q}+\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(2\chi_{l,p,q}+3)l!m!(2p+1)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}^3}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q$$
$$\times\ {}_4F_3\left(\eta_{l,p}+\frac{5}{6},\eta_{l,p}+\frac{7}{6},\eta_{l,p}+\frac{9}{6},\omega_{l,p,q}+\frac{5}{2}\right.$$
$$\left.;\frac{3}{2},\chi_{l,p,q}+\frac{3}{2},\chi_{l,p,q}+2;-\frac{27\Lambda^2}{256\tilde{u}}\right), \tag{2.22}$$

$$\frac{\partial a_D}{\partial u}=\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{1}{2})\Gamma(\omega_{l,p,q}+\frac{1}{2})}{\Gamma(2\chi_{l,p,q}+1)l!m!(2p)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q$$
$$\times\sum_{n=0}^{\infty}\frac{(\eta_{l,p}+\frac{1}{6})_n(\eta_{l,p}+\frac{1}{2})_n(\eta_{l,p}+\frac{5}{6})_n(\omega_{l,p,q}+\frac{1}{2})_n}{(\frac{1}{2})_n(\chi_{l,p,q}+\frac{1}{2})_n(\chi_{l,p,q}+1)_n n!}\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)^n$$
$$\times\left[\ln\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)+\psi_n(\eta_{l,p}+\frac{1}{6})+\psi_n(\eta_{l,p}+\frac{1}{2})+\psi_n(\eta_{l,p}+\frac{5}{6})\right. \tag{2.23}$$
$$\left.+\psi_n(\omega_{l,p,q}+\frac{1}{2})-\psi_n(\frac{1}{2})-\psi_n(\chi_{l,p,q}+\frac{1}{2})-\psi_n(\chi_{l,p,q}+1)-\psi_n(1)\right]$$
$$-\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\left(\frac{\Lambda A}{\tilde{u}^2}\right)\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{5}{2})\Gamma(\omega_{l,p,q}+\frac{5}{2})}{\Gamma(2\chi_{l,p,q}+3)l!m!(2p+1)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q$$
$$\times\sum_{n=0}^{\infty}\frac{(\eta_{l,p}+\frac{5}{6})_n(\eta_{l,p}+\frac{7}{6})_n(\eta_{l,p}+\frac{9}{6})_n(\omega_{l,p,q}+\frac{5}{2})_n}{(\frac{3}{2})_n(\chi_{l,p,q}+\frac{3}{2})_n(\chi_{l,p,q}+2)_n n!}\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)^n$$
$$\times\left[\ln\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)+\psi_n(\eta_{l,p}+\frac{5}{6})+\psi_n(\eta_{l,p}+\frac{7}{6})+\psi_n(\eta_{l,p}+\frac{9}{6})\right.$$
$$\left.+\psi_n(\omega_{l,p,q}+\frac{5}{2})-\psi_n(\frac{3}{2})-\psi_n(\chi_{l,p,q}+\frac{3}{2})-\psi_n(\chi_{l,p,q}+2)-\psi_n(1)\right],$$

where

$$\eta_{l,p}=\frac{l}{3}+\frac{p}{3},\quad \omega_{l,p,q}=l+3p+2q,\quad \chi_{l,p,q}=\frac{l}{2}+p+\frac{q}{2}. \tag{2.24}$$

By analytic continuation and by integration with respect to $\tilde{u}$, we obtain a around the conformal point in the form:

$$a=-2u^{\frac{1}{2}}\sum_{l,p,q}\frac{2^{\chi_{l,p,q}+1}}{l!(2p)!q!3^{\eta_{l,p}}}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}^3}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{-\eta_{l,p}}$$

$$
\begin{aligned}
&\times\Bigg\{\frac{c_1\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{1}{3})\Gamma(\eta_{l,p}+\frac{1}{6})}{\Gamma(\frac{1}{3}-\eta_{l,p})\Gamma(\frac{1}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{5}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{1}{6}}\Psi\left(\frac{1}{6},\frac{1}{6};\frac{1}{3},\frac{2}{3},\frac{5}{3}\right)\\
&\quad+\frac{c_2\Gamma(-\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p})\Gamma(\eta_{l,p}+\frac{1}{2})}{\Gamma(-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{1}{2}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{1}{2}}\Psi\left(\frac{1}{2},\frac{1}{2};\frac{2}{3},\frac{4}{3},2\right) \qquad (2.25)\\
&\quad+\frac{c_3\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}-\frac{1}{3})\Gamma(\eta_{l,p}+\frac{5}{6})}{\Gamma(-\eta_{l,p}-\frac{1}{3})\Gamma(\chi_{l,p,q}-\eta_{l,p}-\frac{1}{3})\Gamma(\chi_{l,p,q}-\eta_{l,p}+\frac{1}{6})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{5}{6}}\Psi\left(\frac{5}{6},\frac{5}{6};\frac{4}{3},\frac{5}{3},\frac{7}{3}\right)\Bigg\}\\
&+\frac{2\pi\sqrt{\pi}\Lambda A}{\tilde{u}^{\frac{1}{2}}}\sum_{l,p,q}\frac{2^{\chi_{l,p,q}+2}}{l!(2p)!q!3^{\eta_{l,p}+2}}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}^3}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{-\eta_{l,p}}\\
&\times\Bigg\{\frac{c_4\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{5}{3})\Gamma(\eta_{l,p}+\frac{5}{6})}{\Gamma(\frac{2}{3}-\eta_{l,p})\Gamma(\frac{2}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{7}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{5}{6}}\Psi\left(\frac{1}{3},-\frac{1}{6};\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)\\
&\quad+\frac{c_5\Gamma(-\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{4}{3})\Gamma(\eta_{l,p}+\frac{7}{6})}{\Gamma(\frac{1}{3}-\eta_{l,p})\Gamma(\frac{1}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{5}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{7}{6}}\Psi\left(\frac{2}{3},\frac{1}{6};\frac{2}{3},\frac{4}{3},\frac{2}{3}\right)\\
&\quad+\frac{c_6\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+1)\Gamma(\eta_{l,p}+\frac{3}{2})}{\Gamma(-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p}+\frac{1}{2})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{3}{2}}\Psi\left(1,1;\frac{4}{3},\frac{5}{3},1\right)\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(a,b;c,d,e) = {}_4F_3\Bigg(&\eta_{l,p}+a,\eta_{l,p}+a+\frac{1}{2},\eta_{l,p}-\chi_{l,p,q}+b,\eta_{l,p}-\chi_{l,p,q}+b+\frac{1}{2} \qquad (2.26)\\
&;c,d,\eta_{l,p}-\omega_{l,p,q}+e;-\frac{256\tilde{u}}{27\Lambda^2}\Bigg),
\end{aligned}
$$

and $c_1=\cdots=c_6=1$. The expression for a_D is given by changing c_i as $c_1=\cot(\eta_{l,p}+\frac{1}{6})\pi,\ c_2=\cot(\eta_{l,p}+\frac{1}{2})\pi,\ c_3=c_4=\cot(\eta_{l,p}+\frac{5}{6})\pi,\ c_5=\cot(\eta_{l,p}+\frac{7}{6})\pi,\ c_6=\cot(\eta_{l,p}+\frac{3}{2})\pi$. If we set $\tilde{L}=\tilde{M}=\tilde{N}=0$, i.e., $A=B=0$ and $\tilde{u}=u'=u-\frac{\Lambda^2}{32}$, we can recover the previous result [21]. As was the case of $N_f=1,2$, the relation $a\sim a_D$ hold on the conformal point, therefore we can recognize the conformal point is the fixed point of this theory.

Let us compare our expressions to the ones obtained by the expansion around the different point from the conformal point. If we consider the massive theory as the generalization from the massless theory, we would treat the bare mass parameter as the deviation from the massless theory. In order to see the behaviour of the field a and a_D in the weak coupling region in this case, we expand the meromorphic differential λ with respect to Λ and mass parameters, and evaluate the integral representation along the corresponding cycle. For

example in the case of $N_f = 1$, λ, a and a_D are given by

$$\begin{aligned}
\lambda &= \frac{x\ dx}{2\pi i\ y}\left(\frac{(x^2-u)}{2(x+m)} - (x^2-u)'\right), \\
a &= \oint_\alpha \lambda, \qquad a_D = \oint_\beta \lambda,
\end{aligned} \tag{2.27}$$

where we use the curve of forth order. The result of the calculation for the field a can be written as

$$\begin{aligned}
a &= \frac{\sqrt{u}}{12\sqrt{\pi}} \sum_{n,l=0} \frac{\Gamma(n-\frac{1}{6})\Gamma(n+\frac{1}{6})\Gamma(l-n)\Gamma(l+3n-\frac{1}{2})}{\Gamma(n+1)\Gamma(-n)\Gamma(3n-\frac{1}{2})\Gamma(l+\frac{1}{2})n!l!}\left(\frac{m^2}{u}\right)^l \left(\frac{-27\Lambda^6}{256u^3}\right)^n \\
&+ \frac{3\sqrt{u}}{32\sqrt{\pi}}\left(\frac{\Lambda^3 m}{u^2}\right) \sum_{n,l=0} \frac{\Gamma(n+\frac{7}{6})\Gamma(n+\frac{5}{6})\Gamma(l-n)\Gamma(l+3n+\frac{3}{2})}{(2n+1)\Gamma(n+1)\Gamma(-n)\Gamma(3n+\frac{3}{2})\Gamma(l+\frac{3}{2})n!l!}\left(\frac{m^2}{u}\right)^l \left(\frac{-27\Lambda^6}{256u^3}\right)^n .
\end{aligned} \tag{2.28}$$

In the massless limit, this expression reduces to the previous result obtained by solving the Picard-Fucks equation [12], which is represented by using the Gauss' hypergeometric function. The expression (2.28) can be verified by expanding the following expression which is represented by using the modular invariant form [21]:

$$\begin{aligned}
\frac{\partial a}{\partial u} &= (-D)^{-\frac{1}{4}} F\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right), \\
\Delta &= -\Lambda^6(256u^3 - 256u^2m^2 - 288um\Lambda^3 + 256m^3\Lambda^3 + 27\Lambda^6), \\
D &= -16u^2 + 12m\Lambda^3.
\end{aligned} \tag{2.29}$$

in the weak coupling region, and by comparing two expressions order by order after u integration. In the $N_f = 2, 3$ case, instead of integrating λ to obtain fields a and a_D, we can evaluate $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ by expanding around the massless point in a similar manner. The results are expressed in terms of the following arguments:

$$\frac{1}{64}\left(\frac{\Lambda^2}{u}\right)^2 \ (N_f = 2), \qquad \frac{1}{256}\left(\frac{\Lambda^2}{u}\right) \ (N_f = 3), \tag{2.30}$$

and appropriate combinations of mass parameters. These are identical to the argument of the hypergeometric function describing the massless theories [12]. These powers of Λ are equivalent to the powers of the instanton term of the curve, and vary as the number of matters we have introduced. On the contrary, the argument of the expression we have

derived in this section is simple compared to (2.29), which is the argument based of the the deviation from the conformal point, and the form of these deviations does not depent on the number of the matters as we have seen in this section. Thus if we use the parametrization from the conformal point, the theory can be described by using the simple deviation from the conformal point even in such case that we discuss the weak coupling behaviour. Furthermore the expression around the massless point in the $N_f = 1, 2$ case can be obtained from our expression for the $N_f = 3$ case by taking suitable double scaling limit to decouple the irrelevant mass parameters. These are obvious advantages to observe the behaviour of the theory by using the expression around the conformal point.

Before closing this section, we discuss the relation between 4-D $SU(2)$ $N = 2$ supersymmetric QCD and 2-D $N = 2$ SCFT, which has been partially analized in our previous paper [21]. Let us review the Landau-Ginzburg description of 2-D $N = 2$ superconformal minimal models with $c = 3$ which describe the torus. Since the theory with central charge $c = 3k/k + 2$ corresponds to the Landau-Ginzburg potential x^{k+2}, we have three type of description; $(k = 1)^3$, $(k = 2)^2$ and $(k = 1)(k = 4)$, as

$$
\begin{aligned}
f_1 &= x^3 + y^3 + z^3, \\
f_2 &= x^4 + y^4 + z^2, \\
f_3 &= x^6 + y^3 + z^2.
\end{aligned}
\tag{2.31}
$$

These are known as the algebraic curve on the (weighted) complex projective space $(W)\mathbf{CP^2}$ with homogeneous coordinates $[x, y, z]$ describing singular torus, and their typical deformation in one parameter ψ are following

$$
\begin{aligned}
\tilde{E}_6 \ &: f = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - \psi_6 xyz = 0, \\
\tilde{E}_7 \ &: f = \frac{x^4}{4} + \frac{y^4}{4} + \frac{z^2}{2} - \psi_7 xyz = 0, \\
\tilde{E}_8 \ &: f = \frac{x^6}{6} + \frac{y^3}{3} + \frac{z^2}{2} - \psi_8 xyz = 0,
\end{aligned}
\tag{2.32}
$$

where we have used appropriate normalization. We can evaluate the period $\mathcal{W}$:

$$
\mathcal{W} = \psi \int_\Gamma \frac{dxdydz}{f}, \tag{2.33}
$$

on each curve in the region $\psi \sim \infty$ by picking up poles of the integrant expanded by $1/\psi$. Altanative approach to obtain the period is solving the Picard-Fuchs equation corresponding to these curves

$$(1-y)y\frac{d^2\mathcal{W}}{dy^2} + (1-2y)\frac{d\mathcal{W}}{dy} - \frac{1}{\alpha}(1-\frac{1}{\alpha})\mathcal{W} = 0, \tag{2.34}$$

where $y = \psi^{-\alpha}$ and $\alpha = 3\ (\tilde{E}_6),\ 4\ (\tilde{E}_7),\ 6\ (\tilde{E}_8)$. As a result, periods are expressed as linear combinations of $F(\frac{1}{\alpha}, 1-\frac{1}{\alpha}, 1; y)$ and $F^*(\frac{1}{\alpha}, 1-\frac{1}{\alpha}, 1; y)$ around $y = 0$ where F is Gauss' hypergeometric function ${}_2F_1$, and F^* is another independent solution corresponding to F. Comparing these results to the expression obtained by setting mass deviations zero in the results we have derived in this section, or the more obvious expression in [21], we find that periods of $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ curves are identical to the periods $\frac{\partial a}{\partial u}$, $\frac{\partial a_D}{\partial u}$ of 4-D $N = 2$ supersymmetric $SU(2)$ QCD with $N_f = 1,\ 2$ and 3 matter fields respectively in the weak coupling region $\tilde{u} \sim \infty$ when the theory has the conformal point. In this way, we can find a simple identification between the moduli parameter of each theory, which is

$$\psi^{\alpha} \longleftrightarrow \tilde{u}, \tag{2.35}$$

up to irrelevant constant factors, and Landau-Ginzburg point $\psi = 0$ of torus corresponds to the fixed point $\tilde{u} = 0$ of N=2 supersymmetric SU(2) QCD. This is another confirmation of our expression around the critical points. It is also interesting that another toric description of torus:

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - \psi zw = 0, \quad \frac{1}{2}z^2 + \frac{1}{2}w^2 - xy = 0, \tag{2.36}$$

can be regarded as the curve which corresponds to $N_f = 0$ curve whose parameter ψ^2 can be identified by deviation from the dyon point.

3 $SU(N)$ pure Yang-Mills theories

In this section we study pure $SU(N)$ theory. In this case the curve is given by

$$y^2 = (x^N - \sum_{i=2}^{N} s_i x^{N-i})^2 - \Lambda^{2N}, \tag{3.1}$$

where s_i $(2 \leq i \leq N)$ are gauge invariant moduli parameter. We treat meromorphic differential λ directly, and calculate the period of meromorphic differential λ, i.e. Higgs field and its dual, which are defined by

$$\lambda = \frac{x}{2\pi i}(x^N - \sum_{i=2}^{N} s_i x^{N-i})' \frac{dx}{y} \tag{3.2}$$

$$a_i = \oint_{\alpha_i} \lambda, \qquad a_D^i = \oint_{\beta_i} \lambda. \tag{3.3}$$

We consider so called Z_N critical point $s_2 = \cdots = s_{N-1} = 0,\ s_N = \pm\Lambda^N$ where the curve becomes [18, 19, 20]

$$y^2 = x^N(x^N \pm 2\Lambda^N). \tag{3.4}$$

First we evaluate the integral in the region $s_i \sim 0$ $(2 \leq i \leq N-1),\ s_N \sim \infty$. To this end, we expand the meromorphic differential λ with respect to Λ^{2N} in the form

$$\lambda = \frac{dx}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})(\Lambda^{2N})^n}{\Gamma(\frac{1}{2})\ n!\ 2n} (x^N - \sum_{i=2}^{N} s_i x^{N-i})^{-2n}. \tag{3.5}$$

Rescaling $x = s_N^{1/N} z$ and $\alpha_i = s_i s_N^{-i/N}$, and expanding with respect to $1/s_N$ and α_i, λ becomes

$$\begin{aligned}\lambda = \frac{s_N^{1/N} dz}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2}) n! 2n} \left(\frac{\Lambda^{2N}}{s_N^2}\right)^n (z^N - 1)^{-2n} \\ \times \sum_{\{m\}}^{\infty} \frac{\Gamma(a_{\{m\}} + 2n)}{\Gamma(2n)} \prod_{i=2}^{N-1} \frac{1}{m_i!} \left(\frac{\alpha_i z^{N-i}}{z^N - 1}\right)^{m_i},\end{aligned} \tag{3.6}$$

where $\{m\} = \{m_2, \cdots, m_{N-1}\}$ and $a_{\{m\}} = \sum_{i=2}^{N-1} m_i$. In order to calculate a_i, we pick up the poles at $e^{\frac{2\pi i k}{N}}$ in meromorphic differential along α_i cycle. By introducing Barnes-type integral representation [22] and multiplying $\sin 2s\pi/\pi$, we integrate from $z=0$ to $z = e^{\frac{2\pi i k}{N}}$ to pick up the poles as

$$\begin{aligned}a_k = s_N^{\frac{1}{N}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_{\{m\}}^{\infty} \int_0^{e^{\frac{2\pi i k}{N}}} dz \frac{\Gamma(s+\frac{1}{2})(-1)^s \Gamma(-s)\Gamma(a_{\{m\}} + 2s)}{\Gamma(\frac{1}{2}) 2s \Gamma(2s)} \frac{\sin 2s\pi}{\pi} \\ \times (z^N - 1)^{-2s - a_{\{m\}}} z^{N a_{\{m\}} - b_{\{m\}}} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{\Lambda^{2N}}{s_N^2}\right)^s,\end{aligned} \tag{3.7}$$

where $b_{\{m\}} = \sum_{i=2}^{N-1} i\, m_i$. Therefore we find that a_k in the region where $s_N \sim \infty$ is given by

$$a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{n,\{m_i\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(n+\frac{1}{2}) \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(\frac{1}{2}) \Gamma(2n+1) n! \Gamma(-2n - b'_{\{m\}} + 1)} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{\Lambda^{2N}}{s_N^2} \right)^n , \tag{3.8}$$

where $b'_{\{m\}} = (b_{\{m\}} - 1)/N$. Note that the phase factor guarantees the constraint $\sum_{i=1}^N a_i = 0$. In order to continuate analytically to the region $s_N \sim \Lambda^N$ and to use various identities, we re-express (3.8) by using the hypergeometric function as

$$a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{n,\{m_i\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} F\left(\frac{b'_{\{m\}}}{2}, \frac{b'_{\{m\}} + 1}{2}; 1; \frac{\Lambda^{2N}}{s_N^2} \right). \tag{3.9}$$

Quite generally the expression of a_k differs by the choice of the branch, therefore we cannot perform analytic continuation of the expression beyond the convergence domain. In the case of $SU(2)$, this process can be justified by comparison to the elliptic singular curve made for torus. For general hyper-elliptic curve, there is no such guarantee for the process. However in our expression, $\frac{\Lambda^{2N}}{s_N^2} = 1$ is the critical point which is just on the boundary of the convergence domain, therefore we can obtain expression around $\frac{\Lambda^{2N}}{s_N^2} = 1$. Performing analytic continuation to $\frac{\Lambda^{2N}}{s_N^2} \sim 1$, and using the identity for the hypergeometric function

$$F(a, b, c, w) = (1 - w)^{-a} F(a, c - b, c, \frac{w}{w - 1}), \tag{3.10}$$

and the quadratic transformation [25]

$$F(2a, 2b, a + b + \frac{1}{2}, z) = F(a, b, a + b + \frac{1}{2}, 4z(1 - z)), \tag{3.11}$$

and also using another identity

$$F(a, b, c, z) = (1 - z)^{c-a-b} F(c - a, c - b, c, z), \tag{3.12}$$

where $z = \frac{1}{2} - \frac{s_N}{2\Lambda^N}$, we can put a_k around the conformal point in the form:

$$\begin{aligned} a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{\{m\}}^{\infty} & \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{s_N + \Lambda^N}{2\Lambda^N} \right)^{\frac{1}{2}} \\ \times & \left\{ \frac{\Gamma(\frac{1}{2} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left(\frac{\Lambda^N + s_N}{s_N} \right)^{-b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; b'_{\{m\}} + \frac{1}{2}; z \right) \right. \\ & \left. + \frac{\Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})} \left(\frac{\Lambda^N + s_N}{s_N} \right)^{-\frac{1}{2}} \left(\frac{s_N - \Lambda^N}{s_N} \right)^{\frac{1}{2} - b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z \right) \right\}. \end{aligned} \tag{3.13}$$

Next we consider a_D^i. In this case we integrate from $z=0$ to $z=e^{\frac{2\pi ik}{N}}$ without multiplying $\sin 2s\pi$ as

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi iN}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\sum_{\{m\}} e^{-2\pi ikb'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})(-1)^{-2s-a_{\{m\}}} \\ \times\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)\Gamma(a_{\{m\}}+2s)\Gamma(-2s-a_{\{m\}}+1)}{\Gamma(\frac{1}{2})\Gamma(2s+1)\Gamma(-2s-b'_{\{m\}}+1)}\left(\prod_{i=2}^{N-1}\frac{\alpha_i^{m_i}}{m_i!}\right)\left(-\frac{\Lambda^{2N}}{s_N^2}\right)^s, \tag{3.14}$$

which is defined modulo a_k in the weak coupling region. We evaluate double poles of this integral and also subtract the contribution from $z=0$ [22] to obtain a_D^k

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi iN}\sum_{\{m\}}^{\infty}\frac{e^{-2\pi ikb'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})}\left(\prod_{i=2}^{N-1}\frac{\alpha_i^{m_i}}{m_i!}\right) \\ \times\sum_{n=0}^{\infty}\frac{(\frac{b'_{\{m\}}}{2})_n(\frac{b'_{\{m\}}}{2}+\frac{1}{2})_n}{\Gamma(n+1)n!}\left(\frac{\Lambda^{2N}}{s_N^2}\right)^n \\ \times\left\{\ln\left(\frac{\Lambda^{2N}}{s_N^2}\right)+\psi(n+\frac{b'_{\{m\}}}{2}) \right. \\ \left. +\psi(n+\frac{b'_{\{m\}}}{2}+\frac{1}{2})-2\psi(n+1)+\pi\cot(b'_{\{m\}}\pi)\right\}. \tag{3.15}$$

Using the analytic continuation to the region $s_N\sim\Lambda$, we have

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi iN}\sum_{\{m\}}\frac{e^{-2\pi ikb'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})}\left(\prod_{i=2}^{N-1}\frac{\alpha_i^{m_i}}{m_i!}\right)\left(\frac{s_N+\Lambda^N}{2\Lambda^N}\right)^{\frac{1}{2}} \\ \times\left\{\frac{\pi\cot(b'_{\{m\}}\pi)\Gamma(b'_{\{m\}}-\frac{1}{2})}{\Gamma(b'_{\{m\}})}\right. \\ \left.\times\left(\frac{s_N+\Lambda^N}{s_N}\right)^{-\frac{1}{2}}\left(\frac{s_N-\Lambda^N}{s_N}\right)^{\frac{1}{2}-b'_{\{m\}}}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2}-b'_{\{m\}};z\right)\right\}. \tag{3.16}$$

Around the critical point, the original roots of the curve e_k^+, e_k^- which both reduce to e_k for $\Lambda=0$, become $e_k^+\simeq\Lambda e^{\frac{2\pi ik}{N}}$, $e_k^-\simeq 0$. The expression (3.13) and (3.16) show that a_k consists of the contribution from both poles whereas a_D^k consists of the contribution from e_k^- which vanishes at the critical point. Of course, we can find an expression for a_D^k which reduces to a_k at the conformal point, i.e., $a_D'^k = a_D^k + a_k$, because a_D^k was defined modulo a_k, which cannot be determined by analytic continuation but by the consistency. Therefore,

around the conformal point a_k and a_D^k behave as

$$a_k \sim a_D^k \sim (s_N - \Lambda^N)^{\frac{N+2}{2N}} + \ const. \tag{3.17}$$

From this result, we recognize that the conformal point is certainly the fixed point of the theory, and the conformal dimension of s_N is $\frac{2N}{N+2}$ [20].

We have used the ordinary type of the analytic continuation but the presence of the factor $\Gamma(-b'_{\{m\}} + \frac{1}{2})$ shows that this factor has poles and the expression (3.13) and (3.16) contain the logarithmic terms. To see this, we decompose $b_{\{m\}}$ mod N as $b_{\{m\}} = Nl + \lambda$ where $l = 0, 1, 2, \cdots$ and $0 \leq \lambda \leq N-1$. Noticing that $b'_{\{m\}} = (b_{\{m\}} - 1)/N$, when N is even and $\lambda = 1 + \frac{N}{2}$, $b'_{\{m\}} - \frac{1}{2}$ becomes integer, thus we find that $\Gamma(-b'_{\{m\}} + \frac{1}{2})$ has poles. That is, around the conformal point of the moduli space of pure $SU(2n)$ theory, there are unstable directions that a_i and a_D^i have logarithmic terms. However except these directions a_i and a_D^i contain no logarithmic term, and since just on the conformal point $\Gamma(-b'_{\{m\}} + \frac{1}{2}) = \Gamma(-\frac{1}{N} + \frac{1}{2})$ there is no logarithmic singularity except $N = 2$, the conformal point is still the fixed point of the theory. When we set $N = 2$, i.e., gauge group $G = SU(2)$, the point we considered is a dyon point. Therefore it is natural that a and a_D have such logarithmic contribution.

As a check of our result and an example, we consider the gauge gourp $G = SU(3)$. We set $u = s_2,\ v = s_3,\ \alpha_2 = u/v^{\frac{2}{3}}$ and $a_{\{m\}} = m,\ b'_{\{m\}} = (2m-1)/3$. In the weak coupling region $v \sim \infty$, our expression reduces to Appell's F_4 system [25] with argument $\frac{4u^3}{27v^2},\ \frac{\Lambda^6}{v^2}$. Analytic continuation to the region $u \sim \infty$ recovers the result in ref.[11] up to the choice of branch for the logarithmic term of a_D^i, which is again represented by Appell's F_4 system. By analytic continuation to around the conformal point, we can find that our expression becomes Horn's H_7 system [25]. To see this, we set $m = 3l + \lambda$ $(l = 0, 1, 2, \cdots,\ \lambda = 0, 1, 2)$ so that a_k and a_D^k are decomposed to $a_k = \sum_{\lambda=0}^{2} a_k^\lambda,\ a_D^k = \sum_{\lambda=0}^{2} a_D^{k\lambda}$. For example, a_k^λ can be written as

$$a_k^\lambda = \frac{v^{\frac{1}{3}} e^{-\frac{2\pi i k}{3}(2\lambda-1)} \sin(\frac{2\lambda-1}{3})\pi\ 2^{\frac{2\lambda-1}{3}}}{i6\pi\Gamma(\frac{1}{2})^3} \left(\frac{u^3}{v^2}\right)^{\frac{\lambda}{3}} \left(\frac{v}{\Lambda^3}\right)^{\frac{1}{2}} \sum_{n,l} \frac{\Gamma(l + \frac{\lambda+1}{3})}{\Gamma(3l + \lambda + 1)} \tag{3.18}$$
$$\times \left\{ \left(\frac{\Lambda^3}{v}\right)^{-\frac{2\lambda}{3}+\frac{5}{6}} \frac{\Gamma(2l + n + \frac{2\lambda}{3} - \frac{1}{3})^2 \sin(\frac{2\lambda}{3} - \frac{1}{3})\pi\ z^n}{\Gamma(2l + n + \frac{2\lambda}{3} + \frac{1}{6}) \sin(\frac{2\lambda}{3} + \frac{1}{6})\pi\ n!} \left(\frac{u^3}{4\Lambda^6}\right)^l \right.$$

$$+\left(\frac{2(v-\Lambda^3)}{v}\right)^{-\frac{2\lambda}{3}+\frac{5}{6}}\Gamma(n+\frac{1}{2})^2\Gamma(2l-n+\frac{2\lambda}{3}-\frac{5}{6})\,\frac{(-z)^n}{n!}\left(\frac{u^3}{(v-\Lambda^3)^2}\right)^l\Bigg\},$$

where $z=\frac{1}{2}(1-\frac{v}{\Lambda^3})$. Because of a factor $\sin(\frac{2\lambda-1}{3})\pi$ in (3.18), the component for $\lambda=2$ disappears, i.e., $a_k^2=0$. For $\lambda=0,1$, the second term can be expressed by Horn's H_7 function as

$$H_7\left(-\frac{5-4\lambda}{6},\frac{1}{2},\frac{1}{2},\frac{2+2\lambda}{3},\frac{u^3}{27(v-\Lambda^3)^2},-\frac{1}{2}(1-\frac{v}{\Lambda^3})\right), \tag{3.19}$$

where $H_7(a,b,c,d,x,y)$ is given by [25]

$$H_7(a,b,c,d,x,y)=\sum_{n,m}\frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m!n!}x^m y^n. \tag{3.20}$$

This means that if we choose the variable $x=\frac{u^3}{27(v-\Lambda^3)^2}$ and $y=-\frac{1}{2}(1-\frac{v}{\Lambda^3})$, Picard-Fuchs equations of the theory should reduce to differential equations of $H_7(a,b,c,d,x,y)$ system which is given by [25]

$$\begin{gathered}\Big\{-y(1+y)\partial_y^2+2x\partial_x\partial_y+(a-1-(b+c+1)y)\partial_y-bc\Big\}H_7=0, \\ \Big\{x(1-4x)\partial_x^2+4xy\partial_x\partial_y-y^2\partial_y^2+(d-(4d+6)x)\partial_x+2ay\partial_y-a(a+1)\Big\}H_7=0,\end{gathered} \tag{3.21}$$

where we have corrected a misprint in ref.[25]. Furthermore, noticing that four independent solutions of this system can be written as

$$\begin{aligned}&H_7(a,b,c,d,x,y)\\ &x^{1-d}H_7(a-2d+2,b,c,2-d,x,y),\\ &y^a\sum_{m,n=0}^{\infty}\frac{(b+a)_{2m+n}(c+a)_{2m+n}}{(d)_m(1+a)_{2m+n}m!n!}(xy^2)^m(-y)^n,\\ &y^{a-2d+2}\sum_{m,n}^{\infty}\frac{(b+a-2d+2)_{2m+n}(c+a-2d+2)_{2m+n}}{(2-d)_m(a-2d+3)_{2m+n}}(xy^2)^m(-y)^n,\end{aligned} \tag{3.22}$$

first and second term of (3.18) with $\lambda=0,1$ correspond to above solutions of this system. Let us chech this point. We start with the Picard-Fuchs equation in this theory for $\Pi=\oint\lambda$ [11]:

$$\begin{aligned}\mathcal{L}_1\Pi&=\Big\{(27\Lambda^6-4u^3-27v^2)\partial_u^2-12u^2v\partial_u\partial_v-3uv\partial_v-u\Big\}\Pi=0,\\ \mathcal{L}_2\Pi&=\Big\{(27\Lambda^6-4u^3-27v^2)\partial_v^2-36uv\partial_u\partial_v-9v\partial_v-3\Big\}\Pi=0.\end{aligned} \tag{3.23}$$

By direct change of variables $x = \frac{u^3}{27(v-\Lambda^3)^2}$ and $y = -\frac{1}{2}(1-\frac{v}{\Lambda^3})$, and some linear combinations of these equations, we can check that the Picard-Fuchs equation (3.23) can be written as

$$x(1-4x)\partial_x^2\Pi_0 - y^2\partial_y^2\Pi_0 + 4xy\partial_x\partial_y\Pi_0 + \frac{2}{3}(1-4x)\partial_x\Pi_0 - \frac{5}{3}y\partial_y\Pi_0 + \frac{5}{36}\Pi_0 = 0,$$
$$y(1+y)\partial_y^2\Pi_0 - 2x\partial_x\partial_y\Pi_0 + \frac{11+12y}{6}\partial_y\Pi_0 + \frac{1}{4}\Pi_0 = 0. \tag{3.24}$$

where $\Pi_0 = y^{-\frac{5}{6}}\Pi$. Comparing this to (3.21), we see that this system is identical to (3.21) with $a = -\frac{5}{6},\ b = c = \frac{1}{2},\ d = \frac{2}{3}$. Substituting these to (3.22), we can find directly that four solutions of the Picard-Fuchs equation of this theory are identical to four functions of the expression (3.18) with $\lambda = 0, 1$, although the first term of (3.18) are not within the Horn's list.

4 $SO(2N)$ pure Yang-Mills theories

In this section we discuss pure $SO(2N)$ theory whose singular points in the strong coupling region are known in arbitrary N [20].

In pure $SO(2N)$ theory the curve and meromorphic differential are given by

$$y^2 = P(x)^2 - \Lambda^{4(N-1)}x^4 = \left(x^{2N} - \sum_{i=1}^{N} x^{2(N-i)}s_i\right)^2 - \Lambda^{4(N-1)}x^4, \tag{4.1}$$
$$\lambda = (2P(x) - xP'(x))\frac{dx}{y}. \tag{4.2}$$

Since the difference from $SU(N)$ theory is only powers of Λ in the instanton correction term, the calculation is almost same as $SU(N)$ theory. What we need is the expression around the point $s_i = 0\,(i \neq N-1),\ s_{N-1} = \pm\Lambda^{2N-2}$ where the curve is degenerate as [20]

$$y^2 = x^{2N+2}(x^{2N-2} \pm 2\Lambda^{2N-2}). \tag{4.3}$$

To this end, it is convenient to evaluate integral in the region $s_i \sim 0\,(i \neq N-1),\ s_{N-1} \sim \infty$. Expanding λ with respect to $\Lambda^{4(N-1)}$ and integrating by part, we can rewrite λ in the following form:

$$\lambda = \int_{-i\infty}^{\infty} \frac{ds}{2\pi i}\frac{dx}{2\pi i}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)}{\Gamma(\frac{1}{2})2s}\left(-\Lambda^{4(N-1)}x^4\right)^s P(x)^{-2s}, \tag{4.4}$$

where we have introduced Barnes-type integral representation as before. Rescaling the variable as

$$x = s_{N-1}^{1/(2N-2)} z = uz, \quad s_i = u^{2i}\alpha_i \, (i \neq N-1). \tag{4.5}$$

and expanding with respect to α_i and Λ^{4N-4}/u^{4N-4}, we have λ in the form:

$$\begin{aligned} \lambda = u \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(s+\frac{1}{2})\Gamma(-s)\Gamma(2s+a_{\{m\}})}{\Gamma(\frac{1}{2})\Gamma(2s+1)} \left(-\frac{\Lambda^{4N-4}}{u^{4N-4}}\right)^s \sum_{\{m\}} \prod_{i\neq N-1}^{N} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} \\ \times \int \frac{dz}{2\pi i} z^{2(N-1)a_{\{m\}}-2b_{\{m\}}} (z^{2N-2}-1)^{-2s-a_{\{m\}}}, \end{aligned} \tag{4.6}$$

where $\{m\} = \{m_1, \cdots, m_{N-2}, m_N\}$ and $a_{\{m\}} = \sum_{i=1, i\neq N-1}^{N} m_i, \quad b_{\{m\}} = \sum_{i=1, i\neq N-1}^{N} i m_i$. In order to obtain a_k, we pick up poles at $z = e^{\frac{2\pi i k}{2N-2}}$ $(0 \leq k \leq N-1)$ along α_k cycle and $z = 0$ along α_N cycle. First we calculate a_k $(0 \leq k \leq N-1)$. To pick up poles at $z = e^{\frac{2\pi i k}{2N-2}}$ we integrate from $z = 0$ to $z = e^{\frac{2\pi i k}{2N-2}}$ multiplying $\sin 2s\pi/\pi$ to find that a_k can be expressed in the form:

$$\begin{aligned} a_k &= \frac{u}{2N-2} \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}}\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!} \left(\frac{\Lambda^{4N-4}}{u^{4N-4}}\right)^n \prod_{i\neq N-1} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} \frac{\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(-2n-b'_{\{m\}}+1)} \\ &= \frac{2u}{2N-2} \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})} \\ &\qquad \times \prod_{i\neq N-1} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} F\left(\frac{b'_{\{m\}}}{2}, \frac{b'_{\{m\}}+1}{2}; 1; \frac{\Lambda^{4N-4}}{u^{4N-4}}\right), \end{aligned} \tag{4.7}$$

where $b'_{\{m\}} = \frac{b_{\{m\}}}{(N-1)} - \frac{1}{(2N-2)}$. The analytic continuation to the region $u^{2N-2} = s_{N-1} \sim \Lambda^{2N-2}$ and the quadratic transformation show that the result is

$$\begin{aligned} a_k = \frac{2u}{2N-2} \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi i b'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})} \prod_{i\neq N-1} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} \\ \times \left[\frac{\Gamma(\frac{1}{2}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}}\right)^{-b'_{\{m\}}} \left(\frac{\Lambda^{2N-2}+u^{2N-2}}{\Lambda^{2N-2}}\right)^{\frac{1}{2}-b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; b'_{\{m\}}+\frac{1}{2}; z\right)\right. \\ +\frac{\Gamma(b'_{\{m\}}-\frac{1}{2})}{\Gamma(b'_{\{m\}})} (1-\frac{\Lambda^{4N-4}}{u^{4N-4}})^{\frac{1}{2}-b'_{\{m\}}} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}}\right)^{b'_{\{m\}}-1} \\ \left. \times \left(\frac{\Lambda^{2N-2}+u^{2N-2}}{\Lambda^{2N-2}}\right)^{b'_{\{m\}}-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}-b'_{\{m\}}; z\right)\right], \end{aligned} \tag{4.8}$$

where $z = \frac{1}{2}(1 - \frac{u^{2N-2}}{\Lambda^{2N-2}})$.

Next we consider a_D^k $(1 \le k \le N-1)$. In this case we integrate meromorphic differential λ from $z = -e^{\frac{2\pi ik}{2N-2}}$ to $z = e^{\frac{2\pi ik}{2N-2}}$ and evaluate douple pole of the integrant without muliplied by $\sin 2s\pi$, and subtract $\frac{1}{2}a_k$ [22]. We have a_D^k in the form:

$$\begin{aligned} a_D^k = \frac{u}{2\pi^2 i} \sum_{n,\{m\}} & \frac{e^{-2\pi ik(b'_{\{m\}})}\Gamma(a_{\{m\}} - b'_{\{m\}})\sin(b'_{\{m\}}\pi)2^{b'_{\{m\}}}}{(2N-2)\Gamma(\frac{1}{2})\Gamma(n+1)^2} \prod_{i \neq N-1}^{N} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} \\ & \times \Gamma(n + \frac{b'_{\{m\}}}{2})\Gamma(n + \frac{b'_{\{m\}}}{2} + \frac{1}{2}) \left(\frac{\Lambda^{4N-4}}{u^{4N-4}}\right)^n \\ \times & \left[\psi(n + \frac{b'_{\{m\}}}{2}) + \psi(n + \frac{b'_{\{m\}}}{2} + \frac{1}{2}) - 2\psi(n+1) + \ln\left(\frac{\Lambda^{4N-4}}{u^{4N-4}}\right) + 2\pi\cot(b'_{\{m\}}\pi)\right]. \end{aligned} \tag{4.9}$$

We make use of the analytic continuation of a_D^k around the conformal point to get

$$\begin{aligned} a_D^k = \frac{2u}{(2N-2)i} & \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi ikb'_{\{m\}}}\Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b_{\{m\}})} \prod_{i \neq N-1} \left(\frac{\alpha_i}{m_i!}\right)^{m_i} \\ & \times \cot(b'_{\{m\}}\pi)\frac{\Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})}(1 - \frac{\Lambda^{4N-4}}{u^{4N-4}})^{\frac{1}{2} - b'_{\{m\}}} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}}\right)^{b'_{\{m\}} - 1} \\ & \qquad \times \left(\frac{\Lambda^{2N-2} + u^{2N-2}}{\Lambda^{2N-2}}\right)^{b'_{\{m\}} - \frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z\right). \end{aligned}$$

As in the pure $SU(N)$ theory, we can claim that $a_k \sim a_D^k$ at the critical point. The behavior of a and a_D near $s_{N-1} = \Lambda^{2N-2},\ s_i = 0$ $(i \neq N-1)$ is

$$a_k \sim a_D^k \sim (s_{N-1} - \Lambda^{2N-2})^{\frac{1}{2} - \frac{1}{2N-2}} + \ const. \tag{4.10}$$

Therefore, we see that the confomal dimension of s_{N-1} is $\frac{2N-2}{N}$ [20].

As was the case of $SU(2n)$, a_i and a_D^i contain the logarithmic terms coming from the factor $\Gamma(\frac{1}{2} - b'_{\{m\}})$ when N of $SO(2N)$ is even, which vanish at the conformal point.

Next we consider a_N and a_D^N. Until now the calculation is same as $SU(N)$ case. However in order to calculate a_N and a_D^N, we have to pick up the pole $x \sim 0$. To this end we rescale the variable of the curve as

$$x^2 = -\frac{s_N}{s_{N-1}}z^2, \quad \beta_i = \frac{s_i}{s_N}\left(-\frac{s_N}{s_{N-1}}\right)^{N-i}, \tag{4.11}$$

where $s_0=-1$, and λ becomes as

$$\lambda=\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\sum_m\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)\Gamma(2s+c_{\{m\}})}{\Gamma(\frac{1}{2})2s\Gamma(2s)}\left(-\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right)^s$$
$$\times\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)z^{4s+2Nc_{\{m\}}-2d_{\{m\}}}(z^2-1)^{-2s-c_{\{m\}}},\qquad(4.12)$$

where $\{m\}=\{m_0,m_1,\cdots,m_{N-2}\}$ and $c_{\{m\}}=\sum_{i=0}^{N-2}m_i,\quad d_{\{m\}}=\sum_{i=0}^{N-2}(N-i)m_i$. By evaluating the line integral from $z=0$ to $z=1$ and by multiplying $\sin 2s\pi/\pi$ to pick up the pole at $z=1$, we get a_N in the region $\frac{s_{N-1}^2}{\Lambda^{4N-4}}>>1$ in the form:

$$\begin{aligned}a_N&=\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{n,\{m\}}\frac{\Gamma(n+\frac{1}{2})\Gamma(2n+d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2n+1)\Gamma((N-1)c_{\{m\}}-d_{\{m\}}+\frac{3}{2})}\left(\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right)^n\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)\\&=2\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{\{m\}}\frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})}\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)\\&\qquad\times F\left(\frac{d_{\{m\}}}{2}+\frac{1}{4},\frac{d_{\{m\}}}{2}+\frac{3}{4};1;\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right).\end{aligned}\qquad(4.13)$$

Notice that this hypergeometic function gives logarithmic term by analytic continuation to the region $\frac{\Lambda^{4N-4}}{s_{N-1}^2}\sim 1$. To see this, we set the variable as

$$y=\frac{\Lambda^{4N-4}}{s_{N-1}^2},\quad z=\frac{\Lambda^{2N-2}-s_{N-1}}{2\Lambda^{2N-2}},\qquad(4.14)$$

and perform the analytic continuation to the region $\frac{s_{N-1}}{\Lambda^{4N-4}}\sim 1$ as

$$\begin{aligned}a_N=&\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{\{m\}}\frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})}\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)\\&\times\Bigg\{(1-y)^{-d_{\{m\}}-\frac{1}{2}}y^{\frac{d_{\{m\}}}{2}-\frac{1}{4}}\frac{\Gamma(d_{\{m\}})}{\Gamma(\frac{d_{\{m\}}}{2}+\frac{1}{4})^2}\sum_{n=0}^{d_{\{m\}}-1}\frac{(\frac{1}{4}-\frac{d_{\{m\}}}{2})_n(\frac{1}{4}-\frac{d_{\{m\}}}{2})_n}{n!(-d_{\{m\}}+1)_n}(1-y)^n\\&+\frac{y^{-\frac{d_{\{m\}}}{2}-\frac{1}{4}}(1-z)^{-d_{\{m\}}}}{\Gamma(\frac{3}{4}-\frac{d_{\{m\}}}{2})\Gamma(\frac{1}{4}-\frac{d_{\{m\}}}{2})\Gamma(d_{\{m\}}+1)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n}{n!(d_{\{m\}}+1)_n}z^n\\&\times\left[\psi(n+1)+\psi(n+d_{\{m\}}+1)-2\psi(n+\frac{1}{2})-\pi-\log(-z)\right]\Bigg\}.\end{aligned}\qquad(4.15)$$

Next we calculate a_D^N. In the region $s_{N-1}\sim\infty$, a_D^N is given by integrating meromorphic differential λ from $z=-1$ to $z=1$ without multiplying $\sin 2s\pi$ and subtracting $\frac{1}{2}a_N$, and

by evaluating double poles as

$$a_D^N = \frac{i s_N^{\frac{1}{2}}}{s_{N-1}^{\frac{1}{2}} 2\pi i} \sum_{n,\{m\}} \frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})} \prod_{i=0}^{N-2} \left(\frac{\beta_i^{m_i}}{m_i!}\right) \frac{(\frac{d_{\{m\}}}{2}+\frac{1}{4})_n (\frac{d_{\{m\}}}{2}+\frac{3}{4})_n}{(n!)^2} y^n$$
$$\times \left[\psi(n+\frac{d_{\{m\}}}{2}+\frac{1}{4}) + \psi(n+\frac{d_{\{m\}}}{2}+\frac{3}{4}) - 2\psi(n+1) + \ln y\right], \qquad (4.16)$$

where $y = \frac{\Lambda^{4N-4}}{s_{N-1}^2}$. Although this logarithmic term disappears by the analytic continuation to the region $s_{N-1} \sim \Lambda^{2N-2}$, another logarithmic term appears

$$a_D^N = \frac{s_N^{\frac{1}{2}}}{2 s_{N-1}^{\frac{1}{2}}} \sum_{n,\{m\}} \frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})} \prod_{i=0}^{N-2} \left(\frac{\beta_i^{m_i}}{m_i!}\right)$$
$$\times \left\{ (1-y)^{-d_{\{m\}}-\frac{1}{2}} y^{\frac{d_{\{m\}}}{2}-\frac{1}{4}} \frac{i\pi\Gamma(d_{\{m\}})}{\Gamma(\frac{d_{\{m\}}}{2}+\frac{1}{4})^2} \sum_{n=0}^{d_{\{m\}}-1} \frac{(\frac{1}{4}-\frac{d_{\{m\}}}{2})_n (\frac{1}{4}-\frac{d_{\{m\}}}{2})_n}{n!(-d_{\{m\}}+1)_n} (1-y)^n \right.$$
$$+ \frac{y^{-\frac{d_{\{m\}}}{2}-\frac{1}{4}} (1-z)^{-d_{\{m\}}}}{\Gamma(\frac{3}{4}-\frac{d_{\{m\}}}{2})\Gamma(\frac{1}{4}-\frac{d_{\{m\}}}{2})\Gamma(d_{\{m\}}+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n z^n}{n!(d_{\{m\}}+1)_n} \qquad (4.17)$$
$$\left. \times \left[\psi(n+1) + \psi(n+d_{\{m\}}+1) - 2\psi(n+\frac{1}{2}) - \log(-z) - \pi\right]\right\}.$$

Thus in $SO(2N)$ theory a_N and a_D^N have the logarighmic terms around this point though the curve become degenerate multiple. Let us consider what is happening. Near $x \sim 0$, α_N cycle and β_N cycle form a small torus, and the curve looks like the curve of pure $SU(2)$ theory. In this case due to our choice of approaching to the point $s_{N-1} = \Lambda^{2N-2}$, $s_N = 0$, this point corresponds to the dyon point for a_N and a_D^N and these have certainly the logarithmic terms. These logarithmic terms are simply caused by the fact that we consider a branch where two of the singularity approach to zero before the theory is going to be at the critical point. This point has been understood in the framework of the $SU(3)$ theory near $u = 0$, $v = \Lambda^2$ [18]. From the expression (4.16) and (4.18), we see that $a_N \sim a_D^N$ on the conformal point. Therefore the existence of logarithmic terms in the expression (4.16) and (4.18) is not harmful.

5 Discussion

We have derived the expression for the periods and Higgs fields and its dual around the conformal point of $SU(2)$ Yang-Mills theory with matter fields, pure $SU(N)$ and pure $SO(2N)$ Yang-Mills theory. In the $SU(2)$ theory with matter fields and the pure $SU(N)$ theory, we have directly recognized the structure of the theories near the conformal points. We find a simple correspondence between the fixed point of 4-D $N = 2$ $SU(2)$ Yang-Mills theory with matter fields and Landau-Ginzburg description of 2-D $N = 2$ SCFT with $c = 3$. For $SU(N)$ and $SO(2N)$ theories we could show a verification of the analytic continuation due to the well known formula of the hypergeometric functions.

It seems interesting that we could obtain the explicit expression of fields around the conformal point even for the theories with higher rank gauge gourps. But the examples we treated in this paper is elementary compared to more complicated varieties of critical points as was shown in [20]. At present, we do not know whether we can find more interesting examples which one can calculate the explicit form of fields. An important question is the verification of the validity of the analytic continuation for these cases, which require further investigation.

Acknowledgment

We would like to thank members of the particle physics group in Hokkaido University for encouragement. One of authers (T.M.) is partially supported by Nukazawa Science Foundation.

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